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Analytical thermal resistances model for eccentric heat source on rectangular plate with convective cooling at upper and lower surfaces

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ABSTRACT

Heat sources on rectangular plate with convective cooling at both upper and lower surfaces are common heat transfer application case in electronic packaging. In this paper, in order to find the analytical solution for this problem, a thermal resistances network model was established based on heat flux flow distribution. The model was used to calculate thermal resistance and predict the mean temperature of the heat source. Simulations by commercial software COMSOL3.5 provided a reference for verification of the model. Data comparisons between simulation and analytical solution show that the network model is accurate to calculate the thermal resistance of the discussion cases and the maximum relative error is 3.47%.

1. Introduction

In electronics packaging, packaging thermal resistance or junction temperatures of dies are the most-used indexes for evaluating the thermal management performance of packaging. Generally, the electronic devices are mounted on a broad printed circuit board. Heat generated by dies conducts through packaging and then transfers onto print circuit board. As dies and packaging usually are much smaller than the substrate or print circuit board where they are located on, the heat dissipation processes can be treated as that heat flux from a portion surface conducts into a larger plate. For these applications, the thermal resistances of print circuit board or heat spreader where chips are bonded on commonly take significant impact on thermal characterization of electronic device or packaging. Therefore, it is critical to find methods to calculate thermal resistance of small heat sources on a larger plate with various boundary conditions.

Thermal spreading resistance takes the majority part of the thermal resistances when heat conducts from small area into a large plate, therefore, most of the calculations in describing such a heat transfer case were based on the concept of thermal spreading resistance. Kennedy [1] studied thermal spreading resistance of uniform heat flux source on a finite cylinder and obtained analytical solutions for a wide range of geometrical parameters with different boundary conditions. Kadambi and Abuaf [2] presented analytical solutions for three-dimensional steady-state and transient thermal conduction with a uniform heat flux on the top surface and a convective heat transfer boundary condition at the bottom. John and Krane [3] obtained the temperature field in the same rectangular geometries as in reference [2] and calculated thermal resistance by using the temperature field. The boundary conditions were based on constant and uniform temperature. Yovanovich [4] and Negus et al. [5] found the solution for the thermal spreading resistance of a single centered heat source on cylinder disk with a heat transfer coefficient at the lower surface. Lee et al. and Song et al. [6,7] analyzed the thermal constriction and spreading resistance in a plate with a uniform heat flux region on one surface and a third kind thermal boundary condition over the other surface and obtained the approximate equation of the thermal constriction and spreading resistance based on the analytical solutions of Yovanovich [4]. Yovanovich et al. did series of studies on thermal spreading resistance for various geometrics [8–10]. Muzychka et al. [11–14] presented analytical solutions of thermal spreading resistance for rectangular flux channels and discussed the influence of geometric and edge cooling. In the models presented in references [8–14], the lower surface of the bodies is convective cooling while the other boundaries are considered to be adiabatic. Since most of Yovanovich's thermal spreading resistance formulas were based on mean

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Nomenclature

a, b, c, d	dimensions of the plate and heat source area, m
A	baseplate area, m^2
A_s	heat source area, m^2
A_0, A_1, A_2, A_3, B_0, B_1, B_2, B_3	Fourier coefficients
h	heat transfer coefficient, $W/(m^2\ K)$
h_u	heat transfer coefficient at upper surface of plate, $W/(m^2\ K)$
h_l	heat transfer coefficient at lower surface of plate, $W/(m^2\ K)$
k	thermal conductivity of plate, W/(m K)
m, n	indices for summations
q	heat flux, W/m^2
R_h	convective film resistance for heat transfer coefficient h, °C/W
R_{h1}	convective film resistance for heat transfer coefficient h at lower surface of cuboid, $\equiv 1/(hcd)$, °C/W
R_{h2}	convective film resistance for heat transfer coefficient h at baseplate area except lower surface of cuboid, $\equiv 1/[h(ab - cd)]$, °C/W
R_{hl}	convective film resistance for heat transfer coefficient h_l, °C/W
R_{hu}	convective film resistance for heat transfer coefficient h_u, °C/W
R_{hu1}	convective film resistance for heat transfer coefficient h_u at lower surface of cuboid, $\equiv 1/(h_u cd)$, °C/W
R_{1D}	one-dimensional thermal conductive resistance of plate, °C/W
R_{1D_cu}	one-dimensional thermal conductive resistance of cuboid, °C/W
$R_{1D\_cu\_u}$	one-dimensional thermal conductive resistance of cuboid in upper spreading layer, °C/W
R_{1D_l}	one-dimensional thermal conductive resistance of lower spreading layer, °C/W
R_s	thermal spreading resistance, °C/W
R'_s	thermal sub-spreading resistance, °C/W
R_{s_l}	thermal spreading resistance of lower spreading layer, °C/W
R_{s_u}	thermal spreading resistance of upper spreading layer, °C/W
R'_{s_u}	thermal sub-spreading resistance of upper spreading layer, °C/W
R_{to}	total thermal resistance of network model, °C/W
R_{to_l}, R_{to_u}	total thermal resistances of plate with lower and upper surface cooling only, respectively, °C/W
t	thickness of plate, m
t_l	thickness of lower spreading layer, m
t_u	thickness of upper spreading layer, m
T	plate temperature, °C
T_f	free flow temperature or ambient temperature, °C
$\overline{T}_s$	mean temperature of heat source, °C
$\overline{T}_{s\_si}$	mean temperature of heat source obtained by simulation, °C
x, y, z	coordinates used for plate, m
x_c, y_c	heat source centroid, m
β	eigenvalues, $\sqrt{\delta^2 + \lambda^2}$
δ	eigenvalues, $n\pi/b$
θ	temperature excess, $\equiv T - T_f$ °C
λ	eigenvalues, $m\pi/a$
ϕ	spreading function
ζ	Dummy variable, m^{-1}

source temperature, Ellison [15] put forward the solution for the maximum thermal spreading resistance in rectangular geometry with one surface convective cooling. Moreover, Ellison provided extensive graphical results which are easily used for engineers.

In most of the above-mentioned references, an adiabatic upper surface has been assumed for calculating thermal spreading resistance. For such an assumption, it is reasonable for cases that the strengths of free convection occurred at both the top and bottom surfaces have great difference. However, there are many cases that convections at both upper and lower surfaces of the plate have nearly the same strength. In these cases, the treatment of adiabatic condition at one surface of the plate will bring unacceptable error for calculating thermal resistance. Hein and Lenzi [16] obtained a solution for an integrated circuit package. In their model, both the chip plane and sink plane are convectively cooled using uniform heat transfer coefficients. Kabir and Ortega [17] analyzed the thermal resistance of a cylindrical substrate with centric heat flux and both surfaces cooling. They presented analytical solution to calculate the thermal resistance of the substrate with both surfaces cooling. Muzychka [18] developed a simple method for predicting discrete heat source temperatures on a finite convectively cooled substrate. By means of influence coefficients, the effect of neighboring source strength and location may be assessed. In this model, the isotropic, orthotropic or compound systems were also considered. The convection in the heat source plane was analyzed too.

In this paper, a thermal resistances network method was used to analyze thermal characterization of eccentric heat on rectangular plate with upper and lower surfaces cooling. The thermal resistances network model was established. The model examination based on comparison with simulation results demonstrates that the network model has good accuracy. The maximum relative error between the results obtained by the network model and simulations is only 3.47% for a random application case.

2. Problem statement

Fig. 1 illustrates an eccentric heat source on isotropic plate with upper and lower surfaces cooling. Here, an eccentric heat source with uniform heat flux is located on an isotropic plate. The upper surface and lower surface of the plate are convectively cooled with heat transfer coefficient h_t and h_l respectively. Sides of the plate are adiabatic. In steady-state situation, the governing equation for describing the heat transfer in Fig. 1 is the Laplace's equation

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{1}$$

where T is the temperature field of the plate at steady-state and it is a function of space as $T(x, y, z)$. Boundary conditions for the system are listed as follows

$$\left.\frac{\partial T}{\partial z}\right|_{z=0} = -\frac{q}{k}, (x,y) \in A_s \tag{2}$$

$$\left.\frac{\partial T}{\partial z}\right|_{z=0} = \frac{h_u}{k}\Big(T(x,y,0) - T_f\Big), (x,y) \in (A - A_s) \tag{3}$$

$$\left.\frac{\partial T}{\partial z}\right|_{z=t} = -\frac{h_l}{k}\Big(T(x,y,t) - T_f\Big), (x,y) \in A \tag{4}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0, \left.\frac{\partial T}{\partial x}\right|_{x=a} = 0 \tag{5}$$

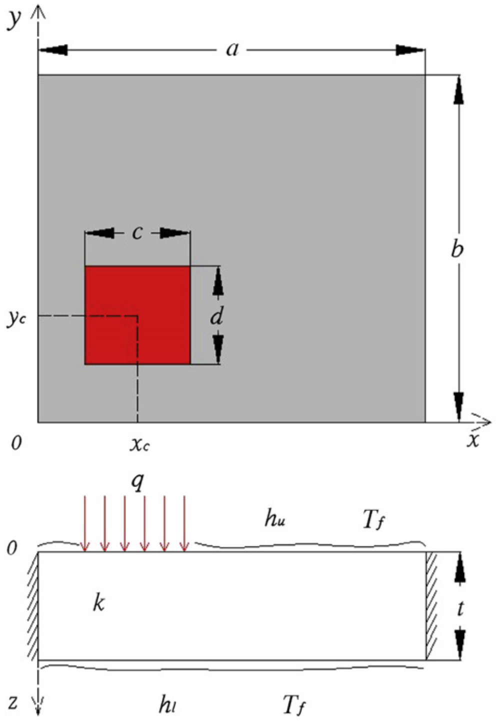

Fig. 1. Eccentric heat source on isotropic plate with upper and lower surfaces cooling.

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0, \quad \left.\frac{\partial T}{\partial y}\right|_{y=b} = 0 \tag{6}$$

A_s is heat source located area. A is the plate's area.

Usually, the separation of variables method is applied to solve partial differential equation (1) with boundary conditions (2)–(6). The solution is assumed to have the form $\theta(x,y,z) = X(x)\cdot Y(y)\cdot Z(z)$. Here $\theta(x,y,z)$ is the temperature excess and it is defined as $T(x,y,z) - T_f$. Eq. (1) and boundary conditions (5), (6) constitute an eigenvalue problem. Therefore, $\theta(x,y,z)$ can be expressed as

$$\begin{aligned}\theta(x,y,z) = {} & A_0 + B_0 z + \sum_{m=1}^{\infty}\cos(\lambda x)[A_1\cosh(\lambda z) + B_1\sinh(\lambda z)] \\ & + \sum_{n=1}^{\infty}\cos(\delta y)[A_2\cosh(\delta z) + B_2\sinh(\delta z)] + \sum_{m=1}^{\infty} \\ & \times \sum_{n=1}^{\infty}\cos(\lambda x)\cos(\delta y)[A_3\cosh(\beta z) + B_3\sinh(\beta z)]\end{aligned} \tag{7}$$

where $\lambda = m\pi/a$, $\delta = n\pi/b$, $\beta = \sqrt{\lambda^2+\delta^2}$, A_i and B_i $(i = 0,1,2,3)$ are the Fourier coefficients. If h_u is zero, taking Fourier expansion at boundary $z = 0$ can obtain coefficients A_i, B_i and finally find the analytical solution as Eq. (7). This is the common way to obtain analytical solutions [2–14]. As h_u is positive, it becomes very difficult to get A_i and B_i by taking Fourier expansion at boundary $z = 0$. The problem will change into a Laplacian problem with mixed boundary conditions. Read [19] once proposed an analytic series method for some of those problems with specific mixed boundary conditions as both Dirichlet and Neumann conditions appeared on one surface. However, this method is still not suitable for solving the mixed boundary condition problem presented in Eqs. (1)–(6). Up till now there are no analytical solutions for the mixed boundary conditions problem described as Eqs. (1)–(6). In the following sections, thermal resistance network method will be used to analyze this problem and resistance network to calculate thermal resistance of the plate with upper and lower surfaces cooling will be presented.

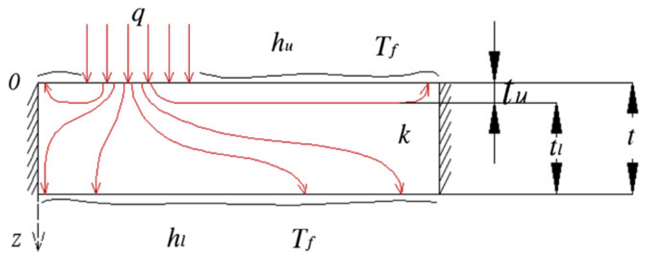

Fig. 2. Heat flows of the system.

3. Network model

3.1. Model description and establishment

For a plate with upper and lower surfaces cooling, heat flux from heat sources spreads into the plate and then transfers to ambient through upper and lower surfaces respectively. Fig. 2 shows the heat flow lines of the system. In the analysis, it is assumed that portion of heat transferring into ambient by the upper surface spreads in an upper layer with thickness of t_u. Similarly, the rest heat, which dissipates into ambient through the lower surface, spreads in a lower layer with thickness t_l.

Thicknesses of upper and lower spreading layers are determined by relative values of the heat transfer coefficients at upper and lower surfaces. In cases that h_u is very small compared to h_l, most of heat will transfer to ambient through the lower surface. As a result, the lower spreading layer will take up a majority of the whole thickness of the plate. Contrarily, if h_l is very small compared to h_u, most of heat will transfer to ambient through the upper surface and the lower spreading layer will take up a majority part of the whole plate. Therefore, t_u and t_l are defined as

$$t_u = \frac{h_u}{h_u + h_l}t \tag{8}$$

$$t_l = \frac{h_l}{h_u + h_l}t \tag{9}$$

Since the plate with both surfaces cooling is divided into upper and lower spreading layers, thermal characterization of both layers should be analyzed firstly. Thermal characterization of the lower spreading layer can be modeled based on the expressions presented in reference [14], where rectangular eccentric heat source is on rectangular plate with only lower surface convective cooling. Thermal characterization of the upper spreading layer will be analyzed by analog with the case of only lower surface cooling in the following.

Fig. 3 shows the schematic of eccentric heat source on plate with lower surface convective cooling and upper surface convective cooling respectively. Before conducting thermal analysis of the plate, two simulations were done by software COMSOL. In the first simulation, the plate's upper surface except heat source area was loaded a heat transfer coefficient 10 W/(m^2 K) while the other surfaces, including sides and lower surfaces, were set as adiabatic. In the second simulation, the lower surface was given a heat transfer coefficient 10 W/(m^2 K) while the other surfaces were adiabatic. Thermal conductivity of the plate was 5 W/(m K) and the ambient temperature was 25 °C in these two simulations. Heat flux

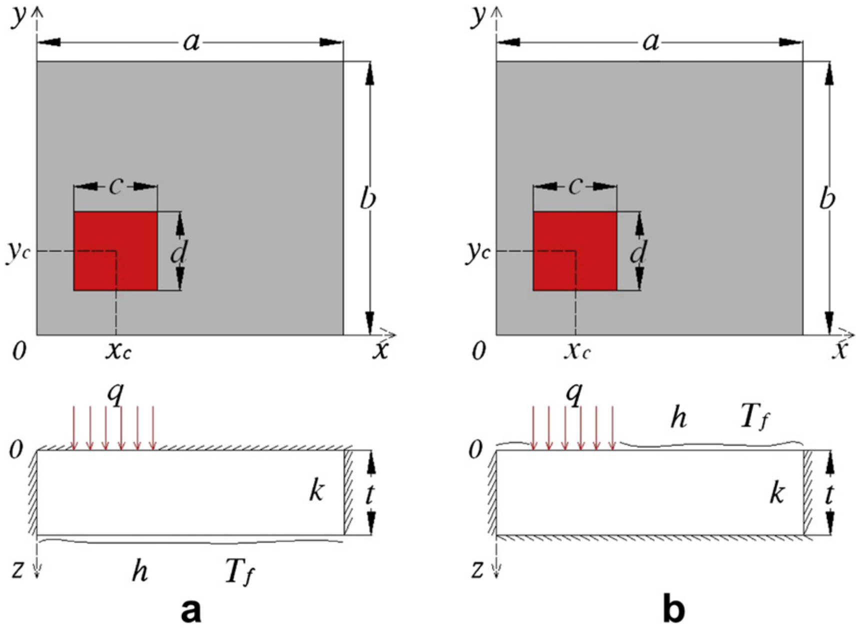

Fig. 3. Schematic of eccentric heat source on plate, (a) lower surface convective cooling, (b) upper surface convective cooling.

in the heat source area was 1667 W/m^2. Heat flow lines in the plate obtained by the two simulations are illustrated in Fig. 4. We define the heat source block as a cuboid whose upper surface is exactly the area of heat source. From Fig. 4, it can be found that heat conducts through the cuboid to the lower surface first and then spread over the plate, finally transfers to ambient from upper surface because of its convective cooling condition. For plate with lower surface cooling, some of heat flow to the lower surface directly in the cuboid and transferred to ambient through lower surface, while the rest heat conducts through parts of the cuboid and then spreads over the plate, finally dissipates to the ambient.

According to the heat flow lines shown in Fig. 4, we can establish thermal resistances networks for plates presented in Fig. 3, which are shown in Fig. 5. $\overline{T}_s$ is mean temperature of the heat source. T_f is ambient temperature. R_s' is defined as thermal sub-spreading resistance. Since the plate is usually thin in electronics packaging, the thermal spreading process happened in the two simulation cases are nearly the same. As a result, R_s' in the two network models in Fig. 5 are treated to be equal. R_{1D_cu} is one-dimensional thermal resistance of the cuboid and it is given as $t/(kcd)$. R_{h1} is the convective film resistance at lower surface of the cuboid while R_{h2} is the one at the rest area of lower surface when the plate is only with lower surface cooling. R_h is convective film resistance at upper surface when the plate is only with upper surface cooling. Because the upper surface of the cuboid is exactly the area of heat flux, R_{h2} equals R_h. r is a constant whose value is in the range 0–1 and it is given as 1/4 in the model. Based on the thermal network model shown in Fig. 5(a), the total thermal resistance of the system with lower surface cooling is expressed as

$$R_{to\_l} = rR_{1D\_cu} + \frac{\left(R_s' + R_{h2}\right)\left[(1-r)R_{1D\_cu} + R_{h1}\right]}{R_s' + R_{h2} + (1-r)R_{1D\_cu} + R_{h1}} \tag{10}$$

with

$$R_{h1} = \frac{1}{hcd}, \quad R_{h2} = \frac{1}{h(ab-cd)} \tag{11}$$

For plate with lower surface cooling, analytical solutions have been found in reference [14] to calculate the total thermal resistance and it is given by

$$R_{to\_l} = \frac{t}{kab} + R_s + \frac{1}{hab} \tag{12}$$

where R_s is the thermal spreading resistance of the system and it is given as

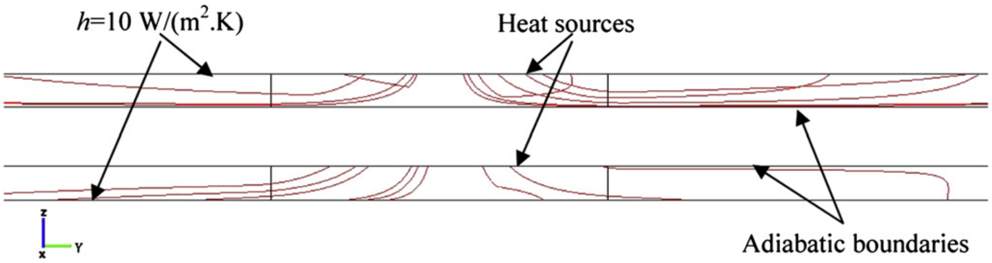

Fig. 4. Heat flow lines in the plate with different cooling boundaries.

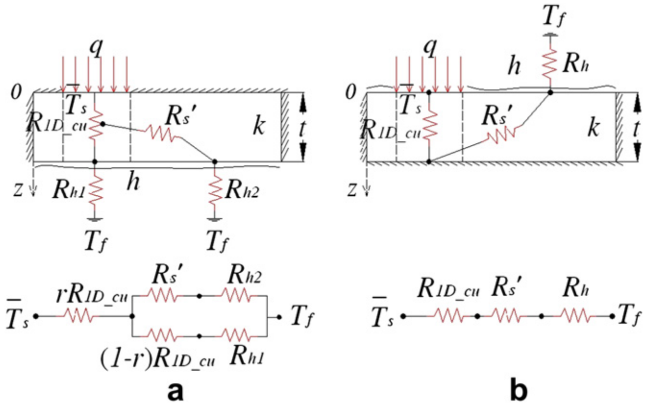

Fig. 5. Thermal network models for cases shown in Fig. 3.

$$
\begin{aligned}
R_s = {} & \frac{8}{abc^2k}\sum_{m=1}^{\infty}\frac{\cos^2(\lambda x_c)\sin^2\left(\frac{c}{2}\lambda\right)}{\lambda^3\phi(\lambda)} \\
& + \frac{8}{abd^2k}\sum_{n=1}^{\infty}\frac{\cos^2(\delta y_c)\sin^2\left(\frac{d}{2}\delta\right)}{\delta^3\phi(\delta)} + \frac{64}{abc^2d^2k}\sum_{m=1}^{\infty} \\
& \times \sum_{n=1}^{\infty}\frac{\cos^2(\lambda x_c)\sin^2\left(\frac{c}{2}\lambda\right)\cos^2(\delta y_c)\sin^2\left(\frac{d}{2}\delta\right)}{\beta\lambda^2\delta^2\phi(\beta)} \qquad (13)
\end{aligned}
$$

with

$$
\phi(\zeta) = \frac{\zeta\sinh(t\zeta) + h/k\cosh(t\zeta)}{\zeta\cosh(t\zeta) + h/k\sinh(t\zeta)} \qquad (14)
$$

where $\lambda = m\pi/a$, $\delta = n\pi/b$, $\beta = \sqrt{\lambda^2+\delta^2}$, and ζ is replaced by λ, δ, β, accordingly. Thus R_s' in Fig. 5 can be obtained as follows,

$$
R_s^{'} = \frac{(R_{to\_l} - rR_{1D\_cu})[(1-r)R_{1D\_cu} + R_{h1}]}{R_{1D\_cu} + R_{h1} - R_{tol}} - R_{h2} \qquad (15)
$$

For plate with upper surface cooling, the total thermal resistance of the plate is expressed as Eq. (16) according to thermal resistances network shown in Fig. 5(b).

$$
R_{to\_u} = R_{1D\_cu} + R_s^{'} + R_h \qquad (16)
$$

As R_{h2} equals R_h, R_s' is the same in Fig. 5(a) and (b), combining Eqs. (15) and (16) can get the final expression for R_{to_u},

$$
R_{to\_u} = R_{1D\_cu} + \frac{(R_{to\_l} - rR_{1D\_cu})[(1-r)R_{1D\_cu} + R_{h1}]}{R_{1D\_cu} + R_{h1} - R_{tol}} \qquad (17)
$$

As stated previously, heat transfer processes in system with both upper and lower surfaces cooling are combination of the ones in plate with single surface cooling, like plates shown in Fig. 3. Fig. 6 shows the heat transfer processes in plate with both upper and lower surfaces cooling and thermal resistances network model is also established based on the heat flow paths.

Previous analysis and results of thermal characterization of single surface cooling plate are applied here to calculate thermal resistance in the network shown in Fig. 6. $R_{1D\_cu\_u}$ is one-dimensional thermal conductive resistance of the cuboid with thickness t_u and it is given as $t_u/(kcd)$. R'_{s_u} is thermal sub-spreading resistance of spreading layer. According to Eq. (15), R'_{s_u} is given by

$$
R_{s\_u}^{'} = \frac{(R_{to\_l\_u} - rR_{1D\_cu\_u})[(1-r)R_{1D\_cu\_u} + R_{hu1}]}{R_{1D\_cu\_u} + R_{hu1} - R_{to\_l\_u}} - R_{hu} \qquad (18)
$$

where $R_{to\_l\_u}$ is total thermal resistance of plate with thickness t_u and h_u at its lower surface, R_{hu1} is $1/(h_ucd)$ and R_{hu} is $1/(h_u(ab-cd))$. Eqs. (12)–(14) are applied to calculate $R_{to\_l\_u}$ by replacing h and t with h_u and t_u, respectively.

R_{1D_l} is one-dimensional thermal conductive resistance of lower spreading layer and is given as

$$
R_{1D\_l} = \frac{t_l}{kab} \qquad (19)
$$

R_{s_L} is thermal spreading resistance of the lower spreading layer. Eqs. (13) and (14) are used to calculate R_{s_L} by replacing h and t with h_l and t_l, respectively. R_{hl} is convective film resistance at lower surface of the plate and it is given by

$$
R_{hl} = \frac{1}{h_lab} \qquad (20)
$$

The total thermal resistance of resistances network shown in Fig. 6 is

$$
R_{to} = R_{1D\_cu\_u} + \frac{\left(R_s^{'} + R_{hu}\right)(R_{1D\_l} + R_{s\_l} + R_{hl})}{R_s^{'} + R_{hu} + R_{1D\_l} + R_{s\_l} + R_{hl}} \qquad (21)
$$

As a result, mean temperature of heat source predicted by the network model is expressed as

$$
\bar{T}_s = qabR_{to} + T_f \qquad (22)
$$

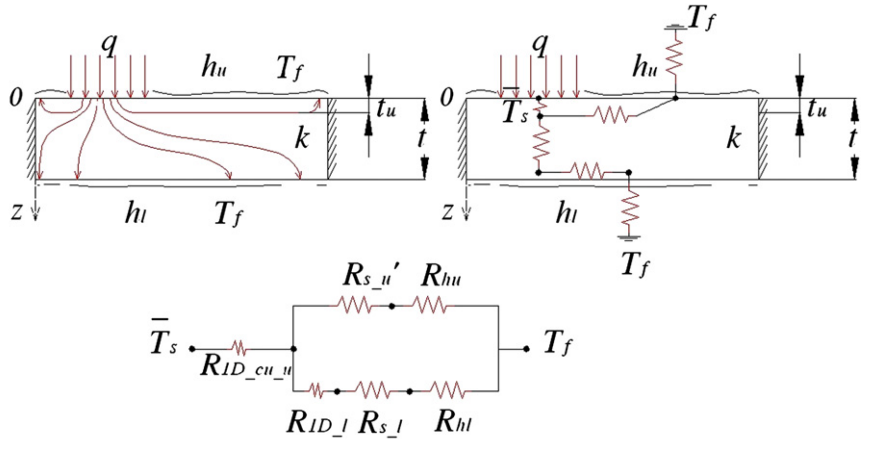

Fig. 6. Heat transfer process and thermal resistances network of plate with upper and lower surfaces cooling.

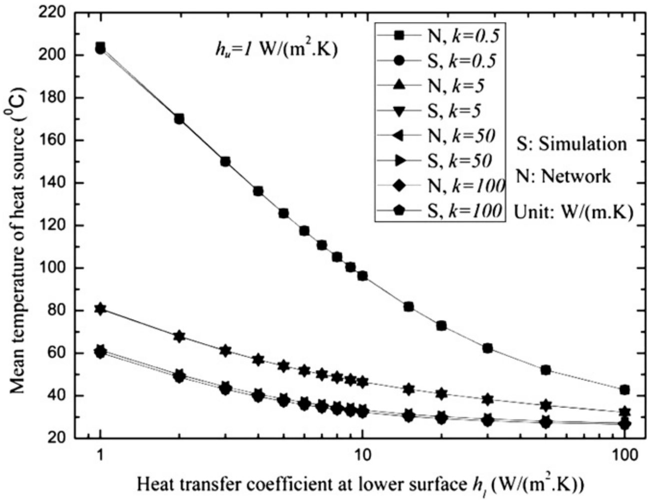

Fig. 7. Data comparisons among network model and simulations as $h_u = 1$ W/(m^2 K).

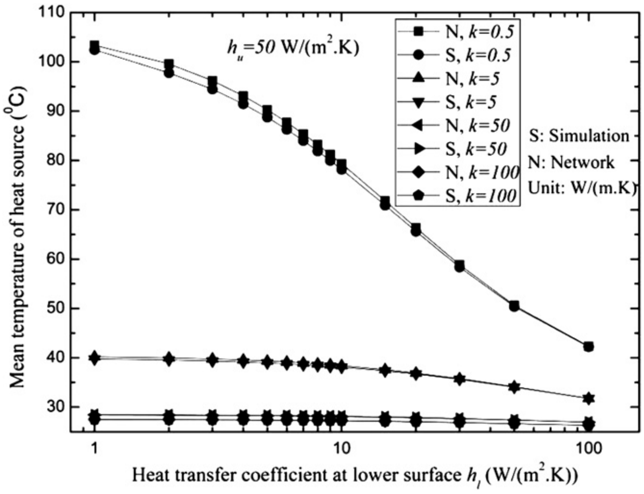

Fig. 9. Data comparisons among network model and simulations as $h_u = 50$ W/(m^2 K).

3.2. Comparisons and discussion

To validate the present network model, random application cases were used to check. In the verification cases, the dimensions of the plate are 150 mm × 100 mm × 2 mm. The size of the heat source is 30 mm by 20 mm and its centroid is (100, 60) mm. The heat flux is 1667 W/(m^2). Thermal conductivity of the plate and heat transfer coefficients at upper and lower surfaces are variable parameters in the calculation. To prove and compare the modeling results, simulations with the same parameters were also done by software COMSOL to provide a reference. Since all the heat transfer processes in the present paper is based on thermal conduction, good simulation result is very reliable for comparison.

As different mesh structures in simulations may lead to different results, the mesh structure was verified to ensure accuracy and avoid unacceptable errors before the computation. The mesh verification was done by comparison between simulation results and analytical solutions presented in reference [14]. In the mesh verification simulations, h_t was set as 0 W/(m^2 K) to meet boundary conditions stated in reference [14]. All other parameters in the simulation were the same as those in the analytical solution. For this case, the plate was meshed by free mesh. Predefined mesh sizes was set as normal and maximum mesh element size of the small cuboid is given by 0.001 m. Using this mesh structure, simulations were done under variable conditions. It is found that mean temperatures of the heat source obtained by simulations and analytical solution are nearly the same while thermal conductivity of the plate k and heat transfer coefficient at lower surface h_l change in wide ranges. The comparison demonstrates that the mesh structure is good enough to offer reliable simulation results.

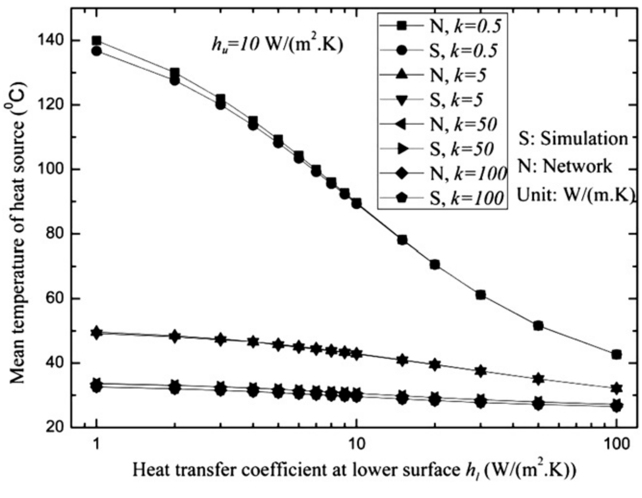

Fig. 8. Data comparisons among network model and simulations as $h_u = 10$ W/(m^2 K).

The data comparisons between the network model and simulations are presented in Figs. 7–9. As shown in these figures, the mean temperatures of heat source predicted by the network model are very close to the ones obtained by simulations at the same conditions. If we defined relative error between network model and simulation as

$$E_r = \frac{\overline{T}_s - \overline{T}_{s\_si}}{\overline{T}_{s\_si} - T_f} \times 100\% \tag{23}$$

where $\overline{T}_s$ is mean temperature of heat source predicted by the network model and $\overline{T}_{s\_si}$ is the one obtained by simulation, the maximum relative error is 3.47%.

4. Conclusions

A thermal resistances network model to calculate thermal resistance of eccentric heat source on rectangular plate with upper and lower surface cooling was established based on heat transfer processes in the plate. A random application case was used verification check of the network model. Simulations were also conducted to offer a reference. The data comparisons show that the network model is able to calculate thermal resistance of eccentric heat source on rectangular plate with upper and lower surface cooling accurately.

Acknowledgments

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