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Examination of the Thermal Cloaking Effectiveness with Layered Engineering Materials *

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The concentrically layered thermal cloaks with isotropic materials could realize the equivalent thermal cloaking effect with Pendry's cloak, while the effectiveness is scarcely investigated quantitatively. Here we examine the cloaking effectiveness quantitatively by evaluating the standard deviation of the temperature difference between the simulated plane with the layered thermal cloak and Pendry's thermal cloak. The design rules for the isotropic materials in terms of thermal conductivity and layer thickness are presented. The present method could quantitatively evaluate the cloaking effectiveness, and could open avenues for analyzing the cloaking effect, detecting the (anti-) cloaks, etc.

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With the birth of transformation optics (TO) by Pendry *et al.* in 2006,[1] its primary outcome (the invisible cloak) has been a fascinating subject and drawn much research interest in the past few years. In the coordinate transformation of TO, Maxwell's electromagnetic (EM) equations keep their form invariant, making it possible to control the EM waves as desires. Inspired by Pendry's work, the heat diffusive equation was found to offer form-invariant characteristic under coordinate transformation, resulting in the birth of transformation thermodynamics concept.[2–7] The thermal cloak, as the typical outcome, can hide an object from heat or render heat invisible in a certain region, realizing many new phenomena in the fields of heat transfer, such as thermal invisibility, thermal bending, heat concentration and energy storage.[8–15] In theory, the realization of the thermal cloak involves with expanding, compressing, rotating or folding of one region/point in one space into another region/point in a different space, resulting in a transformed domain with anisotropic, inhomogeneous, or even extreme material parameters. Such unique materials could realize the perfect thermal cloaking effect, but usually fail to exist in nature or are hard to fabricate in practice.[2,3]

Other than the advances made by theorists, experimentalists also realized thermal cloaks with concentric rings by alternating layered isotropic materials. This is an effective way to realize equivalent thermal cloaking effect without utilizing the thermal metamaterials, though the cloaking effect is slightly weakened or limited. After all, it is a fact that the material parameters of the layered engineering materials are not as anisotropic/extreme as those of the thermal metamaterials. Many studies have been focused on such an alternative method so far, while the criteria to evaluate the cloaking effect of layered thermal cloak are still ambiguous. Only a few studies focus on the examination of thermal cloaking effectiveness so far. Petiteau *et al.*[16] performed a spectral analysis on the cloaking effectiveness of several theoretic thermal cloaks by evaluating the standard deviation of the isotherms in the temperature fields. Here rather than from the perspective of frequency analysis, we present intuitive analyses on the thermal cloaking effectiveness of the layered engineering materials in terms of variations of layer thickness, thermal conductivity, etc.

According to the second law of thermodynamics, heat flow spontaneously diffuses from a region of high temperature to a region of low temperature. For a steady-state heat conduction without heat source, the temperature is governed by Fourier's law $\nabla\cdot(\kappa\nabla T)=0$, where κ is the thermal conductivity, and T is the temperature. In transformation thermodynamics, the heat conduction equation preserves its form before/after a coordinate transformation, i.e., $\nabla'\cdot(\kappa'\nabla' T')=0$. The transformed thermal conductivity κ' is calculated as[2]

$$\kappa'=\frac{J\kappa J^{\rm T}}{\det(J)},\tag{1}$$

where J is the Jacobian transformation matrix between the transformed and original coordinates, $J^{\rm T}$ is the transposed matrix of J, and $\det(J)$ is the determinant of J.

For a two-dimensional Pendry's transformation,[1] as shown in Fig. 1(a), a circular region ($r\le R_2$) in the original space (r,θ,z) is compressed into an annular region ($R_1\le r'\le R_2$) in the transformed space (r',θ',z'), while no transformation occurs along the directions of θ' and z'. The Pendry's transformation mapping can be expressed as

$$r'=\frac{R_2-R_1}{R_2}r+R_1,\quad \theta'=\theta,\quad z'=z,\tag{2}$$

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where R_1 and R_2 are the interior and exterior radii of the annular thermal cloak, and r varies in the range of $[0, R_2]$, while r' varies in the range of $[R_1, R_2]$. The mapping relationship in Eq. (2) indicates that the region outside the annular region ($r = r' > R_2$) remains unchanged, and a cavity ($r' < R_1$) forms in the transformed space in which any object inside the cavity is undetectable. Assuming that the original medium before the transformation is homogeneous and isotropic with constant unit thermal conductivity ($\kappa = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ W/mK), we can obtain the transformed conductivity in the annular region by substituting Eq. (2) into Eq. (1) yielding

$$\kappa' = \begin{bmatrix} \kappa'_{rr} & \kappa'_{r\theta} \\ \kappa'_{\theta r} & \kappa'_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \frac{r'-R_1}{r'} & 0 \\ 0 & \frac{r'}{r'-R_1} \end{bmatrix}. \tag{3}$$

It is seen from Eq. (3) that the thermal conductivities only vary with the change of radial coordinate r'. Along with the increase of r', the radial component κ'_{rr} increases while the tangential component $\kappa'_{\theta\theta}$ decreases monotonously. Here κ'_{rr} varies in the range of $[0, \frac{R_2-R_1}{R_2}]$, and $\kappa'_{\theta\theta}$ in the range of $[\frac{R_2}{R_2-R_1}, +\infty]$ when $r' \in [R_1, R_2]$. For the case of $R_1 = 1$ m and $R_2 = 2$ m, the variations of κ'_{rr} and $\kappa'_{\theta\theta}$ are shown in Fig. 2. It is seen that $\kappa'_{\theta\theta}$ is larger than κ'_{rr}, resulting in the fact that the heat flow in the annular region tends to diffuse along the tangential direction with a higher thermal conductivity (lower thermal resistance). Further, one may discover that these two components are reciprocal to each other and singularity exists in $\kappa'_{\theta\theta}$ at $r' = R_1$. Such unique distribution of thermal conductivity makes it very difficult, if not impossible, to fabricate such metamaterials with natural materials.

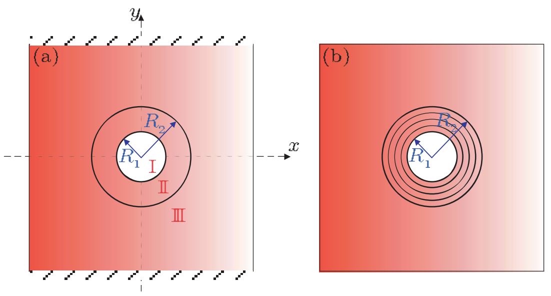

Fig. 1. (Color online) Schematics of (a) Pendry's thermal cloak and (b) layered thermal cloak. Red and white colors denote high and low temperatures, respectively.

From Eq. (3), κ'_{rr} and $\kappa'_{\theta\theta}$ are independent of θ, which indicates that for a ring layer with an infinitesimal thickness, κ'_{rr} and $\kappa'_{\theta\theta}$ keep the same. Inspired by this characteristic, as shown in Fig. 1(b), an alternative method to realize the thermal cloaking effect turns to the concentrically layered materials. When the thicknesses are d_A and d_B (ratio denoted as $d_{B/A} = d_B/d_A$) and constant thermal conductivities are κ_A and κ_B, the corresponding effective radial and tangential thermal conductivities are[6]

$$\kappa_{er}^{-1} = \frac{1}{1+d_{B/A}}(\kappa_A^{-1} + d_{B/A}\kappa_B^{-1}),$$
$$\kappa_{e\theta} = \frac{1}{1+d_{B/A}}(\kappa_A + d_{B/A}\kappa_B). \tag{4}$$

One can first divide the thermal cloak structure into concentric rings along the radial direction, and then alternatively assign the concentric rings with materials A and B, respectively. When the thickness ratio $d_{B/A}$ tends to zero, both κ'_{er} and $\kappa'_{e\theta}$ equal to κ_A, implying a homogenous cloak with material A. When $d_{B/A}$ tends to infinity, both κ'_{er} and $\kappa'_{e\theta}$ equal to κ_B, implying a homogenous cloak with material B. To achieve the equivalent cloaking effect of the layered thermal cloak as Pendry's cloak, we need to make sure that the effective radial and tangential thermal conductivities κ'_{er} and $\kappa'_{e\theta}$ equal to the diagonal components κ'_{rr} and $\kappa'_{\theta\theta}$, respectively, yielding

$$\kappa_A = \frac{1}{2}[(d_{B/A}+1)\kappa'_{\theta\theta} - (d_{B/A}-1)\kappa'_{rr} + \sqrt{\varDelta}]$$
$$\kappa_B = \frac{1}{2d_{B/A}}[(d_{B/A}+1)\kappa'_{\theta\theta} + (d_{B/A}-1)\kappa'_{rr} - \sqrt{\varDelta}], \tag{5}$$

with

$$\varDelta = [(d_{B/A}+1)\kappa'_{\theta\theta}]^2 + [(d_{B/A}-1)\kappa'_{rr}]^2 - 2\kappa'_{rr}\kappa'_{\theta\theta}(d_{B/A}^2+1). \tag{6}$$

This is the rule to design the layered thermal cloak that offers the equivalent cloaking effect as Pendry's cloak with two isotropic materials. When the thickness ratio $d_{B/A}$ equals to 1, the thermal conductivities of materials A and B could be simplified to $\kappa_{A,B} = \kappa'_{\theta\theta} \pm \sqrt{\kappa'^{2}_{\theta\theta} - \kappa'_{rr}\kappa'_{\theta\theta}}$. Further, the product of κ_A and κ_B satisfies $\kappa_A\kappa_B = \kappa'_{rr}\kappa'_{\theta\theta} = 1$. These relations provide a simple rule to design the thermal conductivities of materials A and B with a uniform thickness in the layered structures.

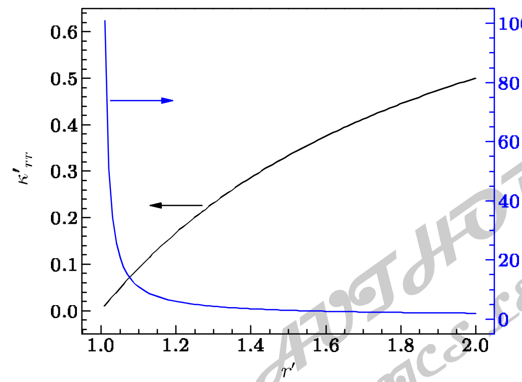

Fig. 2. (Color online) Variations of κ'_{rr} and $\kappa'_{\theta\theta}$ with r'.

To validate the analyses, we employed the COMSOL Multiphysics software to simulate the steady-state heat diffusion process in a 6 m×6 m square plane with/without thermal cloaks. The left and right boundaries were kept at the constant temperatures of 400 K and 300 K, respectively. Other boundaries were insulated. The radii of the annular thermal cloak were 1 m and 2 m, respectively. For Pendry's thermal cloak, the thermal conductivities, i.e., κ'_{rr} and $\kappa'_{\theta\theta}$, were assigned according to Fig. 2. For the layered thermal cloak, the thermal conductivities were calculated according to Eq. (5) at a given thickness ratio $d_{\rm B/A}$ and a certain $\kappa'_{\theta\theta}/\kappa'_{rr}$. Without heat sources or heat sink in the simulated plane, heat diffuses from left to right spontaneously and steady-state temperature fields are achieved.

The temperature profiles of the simulated plane without/with Pendry's thermal cloak are illustrated in Fig. 3. The black curves in the temperature field denote the isotherms. For a homogenous plane, heat tends to diffuse uniformly along the x-axis from the left to the right and the isotherms are straight lines that are orthogonal to the x-axis (direction of heat flow). For a plane with Pendry's thermal cloak, heat tends to travel around the inner circular region and realizes an isothermal region in the central domain. This implies that any object placed in the inner region appears to be concealed since the outside heat flow looks like there is no disturbance at all in the central region. This is the exact cloaking effect of Pendry's thermal cloaks.

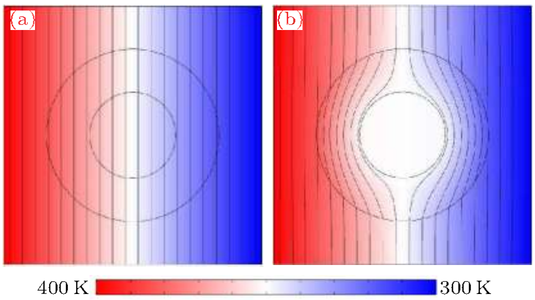

Fig. 3. (Color online) Temperature profiles of (a) homogeneous plane, (b) the simulated plane with Pendry's thermal cloak and (c) with isotropic thermal cloak.

In contrast, the temperature profiles of the simulated plane with layered thermal cloaks are shown in Fig. 4. Three thickness ratios $d_{\rm B/A}$ and four $\kappa'_{\theta\theta}/\kappa'_{rr}$ are considered. Provided with each $d_{\rm B/A}$ and $\kappa'_{\theta\theta}/\kappa'_{rr}$, the thermal conductivities of materials A and B are calculated according to Eq. (5), as listed in Table 1. From Table 1, the increase of $\kappa'_{\theta\theta}/\kappa'_{rr}$ leads to the increase of $\kappa_{\rm A}$ and the decrease of $\kappa_{\rm B}$, while the increase of $d_{\rm B/A}$ results in the increase of both $\kappa_{\rm A}$ and $\kappa_{\rm B}$. It is seen from Fig. 4 that the increases of $\kappa'_{\theta\theta}/\kappa'_{rr}$ and $d_{\rm B/A}$ make heat tend to be away from the inner region and travel close to the outside edge of the thermal cloak. In comparison of the insets in Fig. 4, it is found that $\kappa'_{\theta\theta}/\kappa'_{rr}$ plays a more dominant role than $d_{\rm B/A}$ on the influence of cloaking effect, implying a major direction in designing material parameters for such configuration. Rather than designing the thickness inefficiently, it is worth well designing the thermal conductivity.

To examine the cloaking effectiveness of layered thermal cloaks quantitatively, we calculated the temperature difference (ΔT) between the simulated plane with layered thermal cloaks $T(x,y)_{\rm Layered}$ and Pendry's thermal cloak $T(x,y)_{\rm Pendry}$, as

$$\Delta T(x,y) = T(x,y)_{\rm Layered} - T(x,y)_{\rm Pendry}. \quad (7)$$

Then we calculate the standard deviation (STD) of the selected N points (like a 100×100 array) on the simulated planes to evaluate the cloaking effectiveness, as

$$\mathrm{STD} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\Delta T_i(x,y) - \overline{\Delta T})^2}. \quad (8)$$

Note that a theoretically perfect layered thermal cloak relative to Pendry's thermal cloak possesses a zero temperature difference, i.e., $\Delta T = 0$, throughout the simulated plane, resulting in a zero STD as well. Thus a smaller STD implies an increasing effectiveness. High effectiveness implies better thermal invisibility performance.

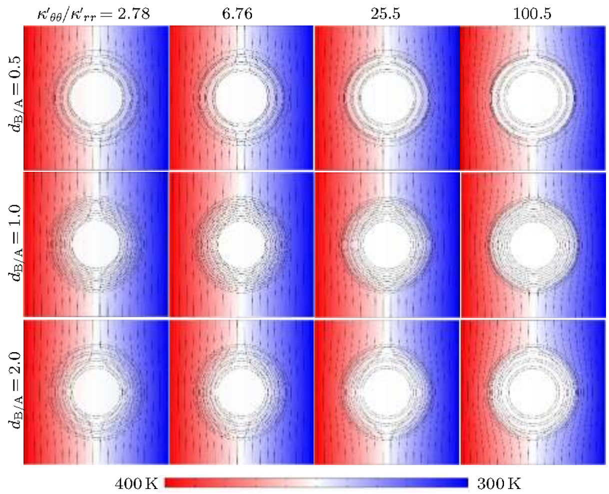

Fig. 4. (Color online) Temperature profiles of the simulated plane with different layered thermal cloaks.

The calculated STD for the twelve layered thermal cloaks in Fig. 4 are presented in Fig. 5. It is seen that when $\kappa'_{\theta\theta}/\kappa'_{rr}$ changes from 2.78 to 100.5, STD increases from 0.56 to 4.50. At certain $d_{\rm B/A}$, STD changes greatly with $\kappa'_{\theta\theta}/\kappa'_{rr}$; while at certain $\kappa'_{\theta\theta}/\kappa'_{rr}$, STD changes slightly with $d_{\rm B/A}$. Both $d_{\rm B/A}$ and $\kappa'_{\theta\theta}/\kappa'_{rr}$ influence STD, while the influence of $\kappa'_{\theta\theta}/\kappa'_{rr}$ seems dominant. The smallest STD corresponds to varied $\kappa'_{\theta\theta}/\kappa'_{rr}$ at different thickness ratios $d_{\rm B/A}$. The reason could be explained by re-examining the temperature profiles in Fig. 4. It is seen that the isotherms in the layered thermal cloaks with small STD are close to those of Pendry's thermal cloak most. A large STD in Fig. 5 corresponds to a major deviation between the isotherms of the layered thermal cloak and Pendry's cloak in Fig. 4. This implies that the STD of the temperature difference is an effective tool to quantitatively evaluate the cloaking effectiveness rather than to examine the isotherms one by one. It is more convenient, precise, and less time-consuming. With this quantitative criterion, we could design the thermal conductivity of materials to effectively tune the heat flow to realize the desired performance.

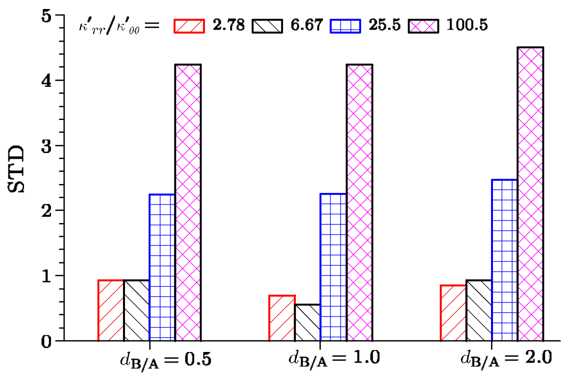

Fig. 5. (Color online) Standard deviations of the temperature difference of the layered thermal cloaks.

Table 1. Thermal conductivities of materials A and B ($\kappa_{\rm A}$, $\kappa_{\rm B}$) in the layered thermal cloaks.

$\kappa'_{\theta\theta}/\kappa'_{rr}$	$d_{\rm B/A}=0.5$	$d_{\rm B/A}=1.0$	$d_{\rm B/A}=2.0$
2.78	(2.38, 0.24)	(3.00, 0.33)	(4.16, 0.42)
6.76	(3.83, 0.14)	(5.00, 0.20)	(7.28, 0.26)
25.5	(7.54, 0.067)	(10.00, 0.10)	(14.89, 0.13)
100.5	(15.02, 0.03)	(20.00, 0.05)	(29.94, 0.07)

In summary, we focus on the examination of the thermal cloaking effectiveness of layered thermal cloak with comparison to Pendry's thermal cloak. In the transformation thermodynamics, we present the design processes of Pendry's thermal cloak (made by thermal metamaterials) and layered thermal cloaks (made by isotropic materials). We calculate the STD of the temperature difference throughout the whole simulated plane with layered or Pendry's thermal cloaks, so as to define the cloaking effectiveness. A smaller STD means closer temperature fields, better thermal cloaking effect, and higher cloaking effectiveness of the layered thermal cloak. This is validated by comparing the isotherms of the simulated temperature fields. It is found that both $\kappa'_{\theta\theta}/\kappa'_{rr}$ and thickness ratio $d_{\rm B/A}$ influence the STD, and the effect of $\kappa'_{\theta\theta}/\kappa'_{rr}$ is dominant. The trend of STD could be used to characterize the closeness of the isotherms in the layered thermal cloaks to those of Pendry's thermal cloak.

This study presents a valid tool to quantitatively evaluate the cloaking effectiveness, and we can apply this method to analyze the temperature fields with/without a thermal cloak, to detect the (anti-) cloaks. Moreover, the present method is not limited to the temperature field, but also could be extended to the electromagnetic field, acoustic field, force field, etc.

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